Homework 1
(120 points)

1. **(20 points) Upper-bound on Entropy.** Let $\Omega = \{1, 2, \ldots, N\}$. Suppose $\mathbb{X}$ is a random variable over the sample space Ω. For shorthand, let us use $x_i = \mathbb{P}[\mathbb{X} = i]$, for each $i \in \Omega$. The entropy of the random variable $\mathbb{X}$ is defined to be the following function.

$$H(\mathbb{X}) := \sum_{i \in \Omega} -p_i \log p_i$$

Use Jensen's inequality on the function $f(x) = \log x$ to prove that the following inequality.

$$H(\mathbb{X}) \leqslant \log N$$

Equality holds if and only if we have $p_1 = p_2 = \cdots = p_N$.

2. **(20 points) Log-sum Inequality.** Let $\{a_1, \ldots, a_N\}$ and $\{b_1, \ldots, b_N\}$ be two sets of non-negative real numbers. Use Jensen's inequality to prove the following inequality.

$$\sum_{i=1}^{N} a_i \log \frac{a_i}{b_i} \geqslant a \log \frac{a}{b},$$

where $a = \sum_{i=1}^{n} a_i$ and $b = \sum_{i=1}^{N} b_i$. Moreover, equality holds if and only if $\frac{a_i}{b_i}$ is equal for all $i \in \{1, \ldots, N\}$.

3. **(20 points) Approximating Square-root.** Our goal is to find a (meaningful and tight) upper-bound for $f(x) = (1-x)^{1/2}$ using a quadratic function of the form

$$g(x) = 1 - \alpha x - \beta x^2$$

Use the Lagrange form of the Taylor's remainder theorem on $f(x)$ around $x = 0$ to obtain the function $g(x)$.

4. **(15+20+5 points) Coupon Collector Problem.** We shall show the following result. "The expected numbers of balls that we need to throw such that every bin has at least one ball is (roughly) $n \log n$."

   (a) Let $i \in \{1, \dots, n-1\}$. Suppose a few balls have already been thrown and we have the guarantee that there are exactly $(i-1)$ bins with > 0 load. Let $\mathbb{N}_i$ be the random variable over the sample space $\{1, 2, \dots\}$ that represents the minimum number of <u>additional balls</u> thrown so that (a total of) i bins have > 0 load. For any $j \in \{1, 2, \dots\}$, prove that

   $$\mathbb{P}\left[\mathbb{N}_i = j\right] = \left(1 - \frac{i-1}{n}\right)\left(\frac{i-1}{n}\right)^{j-1}$$

   (b) Prove that

   $$\mathbb{E}\left[\mathbb{N}_i\right] = \frac{n}{n-i+1}$$

   (c) Denote $\mathbb{N} = \mathbb{N}_1 + \mathbb{N}_2 + \cdots + \mathbb{N}_n$. So, the random variable $\mathbb{N}$ represents the number of balls so that all bins have load > 0. Prove that

   $$\mathbb{E}\left[\mathbb{N}\right] = n\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

   This quantity is roughly $n \log n$. So, we have proven the bound on the coupon collector problem, which was our goal!

5. **(7+7+6 points) Poisson Distribution.** We can also approach the Coupon Collector problem using the Poisson Approximation Theorem. To get us started towards that, let us consider the following problem.

   Suppose we have m balls and n bins. Let $\mathbb{X}$ be the Poisson distribution with mean $\mu = m/n$. That is, for all $i \in \{0, 1, 2, \dots\}$, we have

$$\mathbb{P}\left[\mathbb{X} = i\right] = \exp(-\mu)\frac{\mu^i}{i!}$$

   (a) Find the following probability

$$\mathbb{P}\left[\mathbb{X} \geqslant 1\right]$$

   (b) Suppose $(\mathbb{X}^{(1)}, \dots, X^{(n)})$ be n independent samples of the random variable $\mathbb{X}$. Find the probability

$$\mathbb{P}\left[\mathbb{X}^{(1)} \geqslant 1, \dots, \mathbb{X}^{(n)} \geqslant 1\right]$$

   (c) Substituting $m = n \log n$, compute the probability

$$\mathbb{P}\left[\mathbb{X}^{(1)} \geqslant 1, \dots, \mathbb{X}^{(n)} \geqslant 1\right]$$

   It is left as an exercise to think on how to proceed from here to prove the bounds on the coupon collector problem.